Definitions: The Derivative and More

Definition: The Average Rate of Change. Suppose f is a function defined on an interval $[a, b]$. Then the **average rate of change of** f over $[a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.

Definition: The Derivative. Suppose f is a function defined on an open interval I containing a number p. Then the **derivative** of f at p, denoted by $f'(p)$, is defined by

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p},$$

provided this limit exists. Since $\lim_{x \to p} g(x) = L$ iff $\lim_{h \to 0} g(p + h) = L$, it follows that an equivalent definition of $f'(p)$ is given by

$$f'(p) = \lim_{h \to 0} \frac{f(p + h) - f(p)}{h},$$

provided this limit exists. In this case, f is said to be **differentiable** at p. If f is differentiable at each number in a set S, then f is said to be differentiable on S.

Definition: Critical Numbers. Suppose c is a number in the domain of a function f.

If $f'(c) = 0$ or $f'(c)$ is undefined, then c is a **critical number** of f.

Definition: Local Extrema. Suppose c is a number in the domain of a function f.

a) If there is an open interval I containing c such that $f(c) \geq f(x)$ for all x in I, then $f(c)$ is **a local maximum value** of f.

b) If there is an open interval I containing c such that $f(c) \leq f(x)$ for all x in I, then $f(c)$ is **a local minimum value** of f.

Definition: Absolute Extrema. Suppose f is a function whose domain contains a number c.

a) If $f(c) \geq f(x)$ for all x in the domain of f, then $f(c)$ is **the absolute maximum value** of f.

b) If $f(c) \leq f(x)$ for all x in the domain of f, then $f(c)$ is **the absolute minimum value** of f.

Definition: Suppose f is a function whose domain contains an interval I.

a) If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ (with $x_1, x_2 \in I$), then f is said to be **increasing** on I.

b) If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ (with $x_1, x_2 \in I$), then f is said to be **decreasing** on I.

Definition: Concavity. Suppose f is differentiable on an open interval I.

a) If f' is increasing on I, then f is **concave upward** on I.

b) If f' is decreasing on I, then f is **concave downward** on I.

Definition: Inflection Points. Suppose f is continuous on the open interval (a, b) with $a < c < b$. If the graph of f is concave upward on the interval (a, c) and concave downward on the interval (c, b), or vice versa, then $(c, f(c))$ is an **inflection point** of f.

DERIVATIVE SIGNIFICANCE

Mean-Value Theorem. Suppose f is continuous on $[a, b]$ and differentiable on (a, b) with $a < b$. Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(Interpretation: For "nice" functions f, there is some point $c \in (a, b)$ where the *instantaneous rate of change of f at c* is the same as the *average rate of change of f over* $[a, b]$.)

INTERVALS OF INCREASE/DECREASE. Suppose f is a function that is is continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.
  If f is increasing on (a, b), then $f'(x) \geq 0$ for all $x \in (a, b)$.
- If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.
  If f is decreasing on (a, b), then $f'(x) \leq 0$ for all $x \in (a, b)$.

INTERVALS OF CONCAVITY. Let f be a function whose domain contains an open interval (a, b).

- If $f''(x) > 0$ for all $x \in (a, b)$, then the graph of f is concave upward on (a, b).
- If $f''(x) < 0$ for all $x \in (a, b)$, then the graph of f is concave downward on (a, b).

FERMAT'S THEOREM. Suppose f is a function defined on an open interval containing the number c. If $f(c)$ is a local maximum or local minimum value of f, then c must be a critical number of f.

First-Derivative Test. Let f be a function whose domain contains the open interval (a, b). Suppose $c \in (a, b)$ is a critical number for f with f continuous at c.

- If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$, then $f(c)$ is a local maximum value of f.
- If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, then $f(c)$ is a local minimum value of f.

(Interpretation: *To classify a critical number c for a function f, determine the sign of $f'(x)$ "just to the left" and "just to the right" of c.*)

Second-Derivative Test. Suppose f is a function that is defined on an open interval containing c and suppose $f''(c)$ exists.

- If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local minimum value of f.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum value of f.

Extreme-Value Theorem. Suppose $a < b$. If f is continuous on $[a, b]$ then there exist numbers $x_{min} \in [a, b]$ and $x_{max} \in [a, b]$ such $f(x_{min}) \leq f(x) \leq f(x_{max})$ for all $x \in [a, b]$.

(Remark: *Each of the numbers x_{min} and x_{max} guaranteed by this theorem must be either a critical number of f in (a, b) or an end-point of* $[a, b]$.)