KEY

Calculus I Rates of Change Worksheet for 3.4

As a group, try to answer the following questions.

Buying Widgemawhosits

1) The ordering and transportation cost C for widgemawhosits is

$200x^{-2}$

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the rate of change of C with respect to x when

a) $x = 10$
b) $x = 15$
c) $x = 20$
d) What do these rates of change imply about increasing order size?
e) If I order widgemawhosits, should I buy in bulk?

$$C'(x) = 100\left(-400x^{-3} + \frac{(x+30)(1) - (x)(1)}{(x+30)^2}\right)$$

$$= \frac{-40{,}000}{x^3} + \frac{3000}{(x+30)^2}$$

a) $C'(10) = -38.125$ thousands dollars / hundred orders

b) $C'(15) = -10.3704$ ''

c) $C'(20) = -3.8$ ''

d) Costs per order goes down as the # of orders increase

e) Yes.

Don't Eat Bagged Spinach

2) A population of 500 E. coli bacteria is introduced onto some spinach and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{t^2}{50+t}\right)$$

where t is measured in hours.

a) How many E. Coli are on the spinach after 2 hours?
b) How fast is the population growing after 2 hours? after 4 hours?

$$P'(t) = 500\left(\frac{(50+t)(2t) - (t^2)(1)}{(50+t)^2}\right)$$

$$= \frac{500(100t + t^2)}{(50+t^2)}$$

a) $P(2) = 500\left(1 + \frac{2^2}{50+2}\right) = 538.462$ E. Coli

b) $P'(2) = 1888.\overline{8}$ E. Coli/hr

$P'(4) = 3151.52$ E. Coli/hr

Clown Days

3) A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to the radius r when r is

a) 1 ft

c) 3 ft

e) What if the balloon can only stretch to have a radius of 5 ft. How fast is it expanding when it pops?

$S'(r) = 8\pi r$

a) $S'(1) = 8\pi$

b) $S'(3) = 24\pi$

e) We don't know how FAST it's expanding but we know the rate of increase of surface area with respect to the radius:

$S'(5) = 40\pi$.

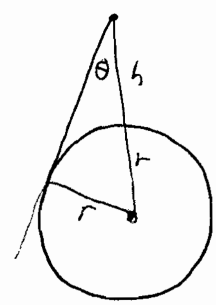
(We will find out how fast later when we get to related rates.)

Satellites

4) When satellites observe the Earth, they can only scan part of the Earth's surface. Some satellites have sensors that can measure the angle θ shown in the figure. Let h represent the satellite's distance from the Earth's surface and let r represent the Earth's radius.

a) Show that $h = r(\csc\theta - 1)$.

b) Find the rate at which the satellite's distance from the Earth's surface is changing with respect to θ when $\theta = \pi/6$. (Assume $r = 3960$ miles).



a) $\sin\theta = \dfrac{r}{h+r}$

$(h+r)\sin\theta = r$

$h + r = \dfrac{r}{\sin\theta}$

$h = r\csc\theta - r$

$h = r(\csc\theta - 1)$

b) $h'(\theta) = 3960(-\csc\theta\cot\theta)$

$h'(\frac{\pi}{6}) = 3960(-2\cdot\sqrt{3}) = 13717.8$ miles/radians

Eggs Thrown Up in the Air

5) Suppose I threw an egg vertically upward in the air, and ran away. Suppose I threw it with an initial velocity of 80ft/s and its height after t seconds is $h(t) = 80t - 16t^2$. Suppose Louie Lou Louie just happened to be walking along and decided to stand right underneath the egg as it was in the air. Suppose Louie Lou Louie is 6 ft tall.

a) What is the maximum height reached by the egg?

b) How many seconds after I throw the egg upwards does it smash onto poor Louie Lou Louie's head?

c) At what speed does it hit Louie Lou Louie on the head?

a) Max. height when $h'(t) = 0$ (changes from pos. velocity (going up) to neg. velocity (going down))

$h'(t) = 80 - 32t$

$0 = 80 - 32t \qquad t = \frac{80}{32}$ sec $= 2.5$ sec.

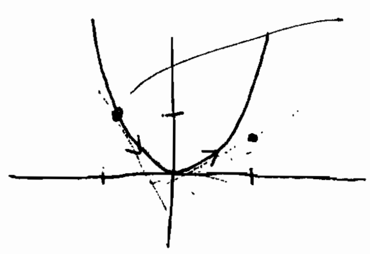
b) $h(t) = 6 = 80t - 16t^2$

$16t^2 - 80t - 6 = 0 \qquad h'(t) \approx 80 - 32\left(\frac{40+\sqrt{1504}}{16}\right) \approx -77.6$ ft/sec

$t = \frac{40+\sqrt{1504}}{16}$ sec

Bonus * I'm a Rabbit in your Headlights**

*) A car is traveling at night along a highway shaped like a parabola with its vertex at the origin (see figure). The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a rabbit located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the rabbit?



$x = -100,\ y = 100$

$y = ax^2$

$100 = a(-100)^2$

$\boxed{\frac{1}{100} = a} \Rightarrow y = \frac{1}{100}x^2$

$f(x) = \frac{1}{100}x^2$

$f'(x) = \frac{2}{100}x =$ slope of tangent line.

2 points on tangent line: $(x, \frac{1}{100}x^2)$ and $(100, 50)$

$$\frac{\frac{1}{100}x^2 - 50}{x - 100} = \frac{1}{50}x$$

$\frac{1}{100}x^2 - 50 = \frac{1}{50}x^2 - 2x$

$0 = \frac{1}{100}x^2 - 2x + 50.$

$0 = \frac{x^2 - 200x + 5000}{100}$

$0 = x^2 - 200x + 5000$

$x = 100 - 50\sqrt{2}$ (by quadratic formula)

$\boxed{\left(100 - 50\sqrt{2}, \frac{1}{100}(100 - 50\sqrt{2})^2\right)}$